RG | Deep Water Waves or Short Waves

7/25/2019

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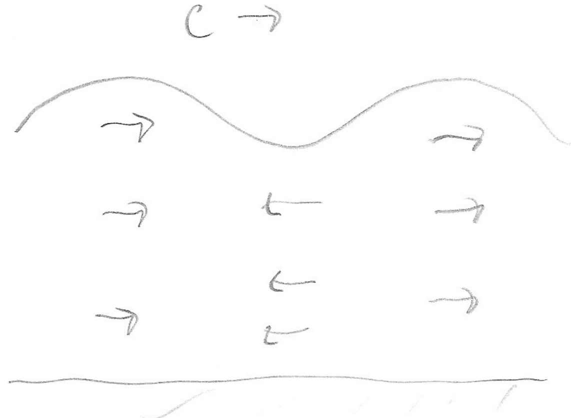
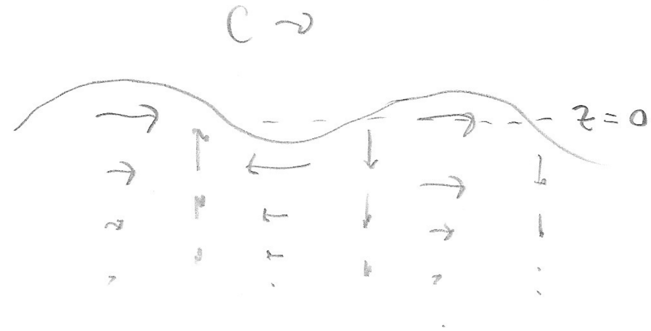
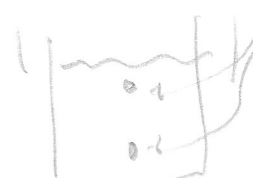




Thought Problem

Drop a bucket:



what do pressure sensors show?

$\rightarrow p_{ATM}$

$$(1) \quad \frac{\partial u}{\partial t} + \frac{1}{\rho_0}\frac{\partial p}{\partial x} = 0$$

$$(2) \quad \frac{\partial w}{\partial t} + \frac{1}{\rho_0}\frac{\partial p}{\partial z} + g = 0$$

$$(3) \quad \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

take derivatives

$$(4) \quad u_{tx} + \frac{1}{\rho} p_{xx} = 0$$

$$(5) \quad w_{tz} + \frac{1}{\rho} p_{zz} = 0$$

$$\left.\begin{matrix}(4)\\(5)\end{matrix}\right\} \quad u_{tx} + w_{tz} + \frac{1}{\rho}\left(p_{xx} + p_{zz}\right) = 0$$

$$(6) \quad u_{xt} + w_{tz} = 0$$

$$(7) \quad p_{xx} + p_{zz} = 0 \qquad \text{Laplace's Eqn.}$$

or $\nabla^2 p = 0$ elliptic eqn.

entire solution is controlled by boundary values.

What if there is a surface wave?

Can it satisfy the equations?

The constraints are the "dispersion relation"

(8) $$p = p_0 e^{i(kx-\omega t)} f(z)$$

put (8) into (7) $\Rightarrow -k^2 f + \underbrace{f''}_{\frac{\partial^2}{\partial z^2}} = 0$

$$f = f_1 e^{kz} + \underbrace{f_2 e^{-kz}}_{0}$$

$$\Rightarrow \boxed{p = p_0 e^{i(kx-\omega t)} e^{kz}}$$

Kinematic Boundary Condition $\frac{\partial \eta}{\partial t} + \cancel{\underline{u} \cdot \nabla \eta} = w(0)$ (14)

linearize: term is $\mathcal{O}(u/c) \ll 1$

Dynamic B.C. (*) $\frac{1}{\rho} p = g\eta$ at $z=0$ (questionable)

from (5) $w_{ttz} - \frac{1}{\rho_0} p_{txx} = 0$

& from (*) $\frac{1}{\rho} p_t = g\eta_t = gw$

$\Rightarrow$ $w_{ttz} - g w_{xx} = 0$ (19)

$\Rightarrow$ $-\omega^2 k W + g k^2 W = 0$ (20)

$w = W e^{i(kx-\omega t)}$

Dispersion Relation

$$\boxed{\omega^2 = gk} \quad (21)$$

So lets talk about

$$\omega^2 = gk \Rightarrow k = \frac{\omega^2}{g} \quad (*)$$

or $\frac{\omega}{k} = \left(\frac{g}{k}\right)^{1/2}$ or $\left(\frac{g\lambda}{2\pi}\right)^{1/2} = C$ phase speed

or $\boxed{C = \frac{g}{\omega} = \frac{gT}{2\pi}}$ useful observationally

and

$$\boxed{\lambda = \frac{g}{2\pi} T^2}$$

$$C = \left(\frac{g}{k}\right)^{1/2} = \left(\frac{g^2}{\omega^2}\right)^{1/2} = \frac{g}{\omega}$$

k from $(*)$

Take a 10 sec. wave (more common on the West coast: the broad shallow shelf on the East coast dissipates the long period waves)

If $kH \sim \mathcal{O}(1)$ you are feeling the bottom.

10 sec wave $\lambda = 156$ m

$C = 15.6$ m/s ~ 30 kts a gale!

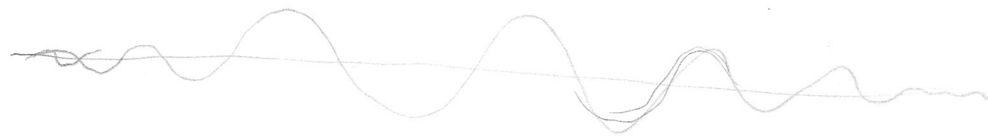
like wind speeds required to create the wave

2 sec wave $\lambda = 6.3$ m

$C = 3.1$ m/sec ~ 6 kts

Group Velocity

wave group:



$$a\cos\alpha + b\cos b = (a+b)\underbrace{\cos\left(\frac{\alpha+\beta}{2}\right)}_{\text{carrier wave}}\underbrace{\cos\left(\frac{\alpha-\beta}{2}\right)}_{\text{modulation (envelope)}}$$

consider 2 waves

$$\eta = (a_1+a_2)\cos\left[\bar{k}(x-\bar{c}t)\right]\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)$$

$$\frac{\Delta\omega}{2} = \frac{\Delta k}{2}\cdot\frac{\partial\omega}{\partial k}$$

$$\eta = (a_1+a_2)\cos\left[\bar{k}(x-\underbrace{\bar{c}}_{\sqrt{g/\bar{k}}}t)\right]\cos\left[\frac{\Delta k}{2}\left(x-\frac{\partial\omega}{\partial k}t\right)\right]$$

and $$\frac{\partial\omega}{\partial k} = \frac{\partial}{\partial k}(gk)^{\frac{1}{2}} = \frac{1}{2}\left(\frac{g}{k}\right)^{\frac{1}{2}} = \frac{1}{2}c \quad ☆$$

$\eta = a\cos[k(x-ct)]$ at the surface

$u = a\omega\cos[\quad]$

$w = a\omega\sin[\quad]$

$u_{max} = ak^{c}c$, max steepness $\left.\frac{\partial\eta}{\partial x}\right|_{max} = ak$